

On the resonant Lane-Emden problem for the p -Laplacian

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Abstract

We study the resonant Lane-Emden problem

$$\begin{cases} -\Delta_p u &= \lambda_p |u|^{q-2} u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where λ_p is the first eigenvalue of the p -Laplacian operator Δ_p , $p > 1$, and q is close to p . The solution u_q of this problem is obtained by a suitable scaling of a positive extremal w_q of the Rayleigh quotient R_q associated with the immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Our main result is the convergence $u_q \rightarrow \theta_p e_p$ in $C^1(\overline{\Omega})$ as $q \rightarrow p$, where

$$\theta_p := \exp \left(\|e_p\|_p^{-p} \int_{\Omega} e_p |\ln e_p| dx \right)$$

and e_p is the positive and L^∞ -normalized first eigenfunction of the p -Laplacian. An immediate consequence of our result is that the best constant of the immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is differentiable at $q = p$. Our approach allow us to handle both sub-linear and super-linear cases simultaneously and it is based on L^∞ and L^q estimates that we prove by using level set techniques. We also generalize to the space $C^1(\overline{\Omega})$ previous results on the asymptotic behavior (as $q \rightarrow p$) of the positive solutions $u_{\lambda,q}$ of the non-resonant problem ($0 < \lambda \neq \lambda_p$). Moreover, we indicate how to obtain the first eigenpair (λ_p, e_p) of the p -Laplacian as the limit (when $q \rightarrow p$) in the space $C^1(\overline{\Omega})$ of a suitable scaling of the pair $(\lambda, u_{\lambda,q})$ for each $\lambda > 0$. For computational purposes the advantage of this approach is that λ does not need to be close to λ_p .

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1 Introduction

Consider the Lane-Emden problem

$$\begin{cases} -\Delta_p u &= \lambda |u|^{q-2} u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

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where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, $N \geq 2$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator with $p > 1$, and $1 < q < p^*$, with p^* denoting the Sobolev critical exponent defined by $p^* = Np/(N-p)$, if $1 < p < N$, and $p^* = \infty$, if $p \geq N$.

If $q = p$, we have the p -Laplacian eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

whose first eigenvalue λ_p is positive, simple, isolated and admits a first positive eigenfunction $e_p \in C^{1,\alpha}(\overline{\Omega})$ satisfying $\|e_p\|_\infty = 1$ in Ω . (We maintain this notation from now on.) Moreover, λ_p also is characterized by the minimizing property

$$\lambda_p = \min \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} = \frac{\int_\Omega |\nabla e_p|^p dx}{\int_\Omega |e_p|^p dx}. \quad (3)$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1) if, and only if,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \lambda \int_\Omega |u|^{q-2} u \varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(\Omega). \quad (4)$$

This means that u is a critical point of the energy functional $J_{\lambda,q} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$J_{\lambda,q}(v) = \frac{1}{p} \int_\Omega |\nabla v|^p dx - \frac{\lambda}{q} \int_\Omega |v|^q dx.$$

In the super-linear case $p < q < p^*$ the existence of at least one positive weak solution $u_{\lambda,q}$ of (1) that minimizes $J_{\lambda,q}$ among all possible nonzero weak solutions follows from standard variational methods. Weak positive solutions satisfying this minimizing property are known as ground states. As shown in [9] nonuniqueness of positive weak solutions occurs for ring-shaped domains when q is close to p^* . On the other hand, when Ω is a ball, there exists only one positive weak solution (see [1]), thus it must be a ground state. For the Laplacian ($p = 2$) uniqueness happens for a general domain Ω if q is sufficiently close to 2 (see [6, 15]).

In the sub-linear case $1 < q < p$ the existence of a positive weak solution follows both from the sub- and super-solution method or from standard variational arguments concerning the global minimum of the energy functional $J_{\lambda,q}$ in $W_0^{1,p}(\Omega)$. The uniqueness of such a weak positive solution is proved in [12].

In both cases a proof of existence by applying the subdifferential method can be found in [16], where we can also find the proof of the boundedness (in the sup norm) of any positive weak solution of (1), a result which implies the $C^{1,\alpha}(\overline{\Omega})$ -regularity by applying well-known estimates (see [8, 14, 18]).

With different goals, the asymptotic behavior when $q \rightarrow p$ of the Lane-Emden problem (1) has been studied by many authors since the 1990s. For example, in [9] for $p > N$, $\lambda = 1$ and $q \rightarrow p^*$; or in [20] for $p = N$, $\lambda = 1$ and $q \rightarrow \infty$. In [11], the asymptotic behavior in $W_0^{1,p}(\Omega)$ of the positive ground states $u_{\lambda,q}$, as $q \rightarrow p^+$, was described for all positive values of λ . In that paper a ground state $u_{\lambda,q}$ was obtained as the minimum of $J_{\lambda,q}$ on the positive Nehari manifold. More recently, the asymptotic behavior with $q \rightarrow p^-$ in $W_0^{1,p}(\Omega)$ was described in [3].

In the present paper we first consider the resonant Lane-Emden problem (1), that is, the case $\lambda = \lambda_p$ and its positive solution u_q , obtained by means of a suitable scaling of the L^q -normalized extremal w_q of the Rayleigh quotient $R_q(u) = \|\nabla u\|_p^p / \|u\|_q^p$. (From now on $\|v\|_r$ stands for $(\int_\Omega |v|^r dx)^{\frac{1}{r}}$.)

By using level set techniques we prove an *a priori* bound in $L^\infty(\Omega)$ for u_q , valid if $q \in [1, p^{\frac{N+1}{N}})$. We apply it to obtain uniform boundedness of these functions in $C^{1,\alpha}(\overline{\Omega})$ and prove the C^1 -convergence $u_q \rightarrow \theta_p e_p$ when $q \rightarrow p$, where

$$\theta_p := \exp \left(\|e_p\|_p^{-p} \int_{\Omega} e_p^p |\ln e_p| dx \right).$$

A consequence of this result is the differentiability at $q = p$ of the function $q \in [1, p^*) \mapsto \lambda_q \in \mathbb{R}$, where λ_q denotes the minimum of the Rayleigh quotient R_q on $W_0^{1,p}(\Omega) \setminus \{0\}$.

We emphasize that our approach unifies both sub-linear and super-linear problems, which is an important contribution to the complete understanding of the asymptotic behavior of the resonant case. (Previous approaches treated separately the sub-linear and super-linear problems and although the families of solutions were known to have a subsequence converging to a multiple of e_p , this multiple was unknown; in principle, different multiples of e_p could be obtained as limits of these families and even these limits could depend on the converging subsequence.)

Regarding the non-resonant case $0 < \lambda \neq \lambda_p$, we prove that

$$\lim_{q \rightarrow p^-} \|u_{\lambda,q}\|_{C^1} = \begin{cases} 0, & \text{if } \lambda < \lambda_p \\ \infty, & \text{if } \lambda > \lambda_p \end{cases} \quad \text{and} \quad \lim_{q \rightarrow p^+} \|u_{\lambda,q}\|_{C^1} = \begin{cases} \infty, & \text{if } \lambda < \lambda_p \\ 0, & \text{if } \lambda > \lambda_p. \end{cases} \quad (5)$$

These results are stronger than those in [3] and [11] and obtained by a different method. (Here $\|v\|_{C^1} := \|v\|_\infty + \|\nabla v\|_\infty$ is the norm of a function v in $C^1(\overline{\Omega})$.)

Finally, we show how to obtain a first eigenpair of the p -Laplacian as limit of positive, normalized (in L^∞ and L^q norms) solutions of (1). More precisely, we prove that for each $\lambda > 0$ and for any $q_n \rightarrow p$ we have the $C^1(\overline{\Omega})$ convergences

$$\lim_{q_n \rightarrow p} \left(\lambda \|u_{\lambda,q_n}\|_\infty^{q_n-p} \right) = \lim_{q_n \rightarrow p} \left(\lambda \|u_{\lambda,q_n}\|_{q_n}^{q_n-p} \right) = \lambda_p, \\ \frac{u_{\lambda,q_n}}{\|u_{\lambda,q_n}\|_\infty} \rightarrow e_p \quad \text{and} \quad \frac{u_{\lambda,q_n}}{\|u_{\lambda,q_n}\|_{q_n}} \rightarrow \frac{e_p}{\|e_p\|_p}.$$

This result might be useful for numerical computation of the first eigenvalue of the p -Laplacian (see [5]), since it does not require λ to be close to λ_p and is also independent of the sequence $q_n \rightarrow p$.

This paper is organized as follows. Section 2 is dedicated to the resonant case. We obtain L^q and L^∞ estimates of the solution u_q of the resonant problem and prove our results on the asymptotic behavior as $q \rightarrow p$. In this section we also show that a suitable scaling of the pair $(\lambda, u_{\lambda,q})$ associated with the non-resonant Lane-Emden problem (1) for arbitrary $\lambda > 0$ produces the first eigenpair (λ_p, e_p) . In Section 3 we handle the asymptotic behavior (5) for the non-resonant problem.

2 Asymptotic behavior for the resonant problem

In this section we consider the resonant Lane-Emden problem

$$\begin{cases} -\Delta_p u &= \lambda_p |u|^{q-2} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \quad (6)$$

and its energy functional given by

$$I_q(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \frac{\lambda_p}{q} \int_{\Omega} |u|^q dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

A well-known consequence of the compactness of the immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < p^*$ is that the quotient $R_q : W_0^{1,p}(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$R_q(u) = \frac{\int_{\Omega} |\nabla u|^p dx}{\left(\int_{\Omega} |u|^q dx\right)^{\frac{p}{q}}}$$

attains a positive minimum at a positive and L^q -normalized function $w_q \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$:

$$\|w_q\|_q = 1 \text{ and } \lambda_q := \inf \left\{ R_q(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} = R_q(w_q). \quad (7)$$

It is straightforward to verify that w_q is a weak solution of

$$\begin{cases} -\Delta_p u &= \lambda_q |u|^{q-2} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

As a consequence of this fact, the function

$$u_q = \left(\frac{\lambda_q}{\lambda_p} \right)^{\frac{1}{q-p}} w_q \quad (8)$$

is a positive weak solution of the resonant Lane-Emden problem (6) and this notation will be maintained from now on.

In the super-linear case $1 < p < q < p^*$ this weak positive solution satisfies the minimizing properties

$$\|u_q\|_q \leq \|v\|_q \quad (9)$$

and

$$I_q(u_q) \leq I_q(v) \quad (10)$$

if $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a weak solution of (6), so u_q is a ground state.

There is no extremal for R_{p^*} in the critical case $q = p^*$. However, the value of λ_{p^*} does not depend on the bounded domain Ω . In fact, $\lambda_{p^*} = S_{N,p}^p$ where $S_{N,p}$ is the *Sobolev constant*, that is, the best constant of the *Sobolev Inequality*

$$S_{N,p} \left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leq \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}, \text{ for all } u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}. \quad (11)$$

It is well-known (see [17]) that

$$S_{N,p} = \sqrt{\pi} N^{\frac{1}{p}} \left(\frac{N-p}{p-1} \right)^{\frac{p-1}{p}} \left(\frac{\Gamma(N/p) \Gamma(1+N-N/p)}{\Gamma(1+N/2) \Gamma(N)} \right)^{\frac{1}{N}} \quad (12)$$

where $\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds$ is the Gamma function.

Therefore, for any bounded domain $D \subset \mathbb{R}^N$ one also has

$$S_{p,N} \left(\int_D |u|^{p^*} dx \right)^{\frac{1}{p^*}} \leq \left(\int_D |\nabla u|^p dx \right)^{\frac{1}{p}}, \text{ for all } u \in W^{1,p}(D) \setminus \{0\}. \quad (13)$$

Lemma 1 *The function $q \in [1, p^*) \rightarrow |\Omega|^{\frac{p}{q}} \lambda_q \in \mathbb{R}$ is non-increasing. In particular*

$$\|u_q\|_q = \left(\frac{\lambda_q}{\lambda_p} \right)^{\frac{1}{q-p}} \leq |\Omega|^{\frac{1}{q}}. \quad (14)$$

Proof. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $1 \leq q_1 < q_2 < p^*$. Hölder inequality implies that

$$\left(|\Omega|^{-1} \int_{\Omega} |u|^{q_1} dx \right)^{\frac{p}{q_1}} \leq |\Omega|^{-\frac{p}{q_1}} \left[\left(\int_{\Omega} |u|^{q_2} dx \right)^{\frac{q_1}{q_2}} |\Omega|^{1-\frac{q_1}{q_2}} \right]^{\frac{p}{q_1}} = \left(|\Omega|^{-1} \int_{\Omega} |u|^{q_2} dx \right)^{\frac{p}{q_2}}.$$

Hence we obtain

$$|\Omega|^{\frac{p}{q_2}} \lambda_{q_2} \leq |\Omega|^{\frac{p}{q_2}} R_{q_2}(u) \leq |\Omega|^{\frac{p}{q_1}} R_{q_1}(u),$$

thus implying

$$|\Omega|^{\frac{p}{q_2} - \frac{p}{q_1}} \lambda_{q_2} \leq R_{q_1}(u) \text{ for all } u \in W_0^{1,p}(\Omega) \setminus \{0\},$$

what yields

$$|\Omega|^{\frac{p}{q_2} - \frac{p}{q_1}} \lambda_{q_2} \leq \lambda_{q_1} = \inf \left\{ R_{q_1}(u) : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}$$

and $|\Omega|^{\frac{p}{q_2}} \lambda_{q_2} \leq |\Omega|^{\frac{p}{q_1}} \lambda_{q_1}$. The inequality in (14) now follows from the monotonicity of $|\Omega|^{\frac{p}{q}} \lambda_q$. \square

Now, inspired by arguments based on level set techniques developed in [13] (see also [4, Theor. 2]) we develop an explicit upper bound to any bounded and positive solution v_q of the resonant Lane-Emden problem (6). This bound will allow us to obtain uniform estimates for $\|u_q\|_{\infty}$ with respect to $q \in [1, p(\frac{N+1}{N})]$.

Theorem 2 *Let $v_q \in W_0^{1,p}(\Omega)$ be a bounded and positive solution of the resonant Lane-Emden (6). Then, for all $1 \leq q < p(\frac{N+1}{N})$ we have*

$$\|v_q\|_{\infty} \leq (K_{N,p})^{\frac{p+N(p-1)}{p+N(p-q)}} (\|v_q\|_1)^{\frac{p}{p+N(p-q)}} \quad (15)$$

where

$$K_{N,p} := \left(\frac{\lambda_p}{S_{N,p}^p} \right)^{\frac{N}{p+N(p-1)}} \left(\frac{p+N(p-1)}{p} \right)$$

and $S_{N,p}$ is given by (12). Thus, $K_{N,p}$ is a positive constant that depends only on p , N and Ω .

Proof. For each $0 < t < \|v_q\|_{\infty}$ let

$$A_t = \{x \in \Omega : v_q > t\}.$$

The function

$$(v_q - t)^+ = \max \{v_q - t, 0\} = \begin{cases} v_q - t, & \text{if } v_q > t, \\ 0, & \text{if } v_q \leq t, \end{cases}$$

belongs to $W_0^{1,p}(\Omega)$. Classical results [8, 14, 18] guarantee that $v_q \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Therefore, we have

$$\int_{A_t} |\nabla v_q|^p dx = \lambda_p \int_{A_t} v_q^{q-1} (v_q - t) dx \leq \lambda_p \|v_q\|_{\infty}^{q-1} \int_{A_t} (v_q - t) dx. \quad (16)$$

(Note that A_t is open and therefore $\nabla(v_q - t)^+ = \nabla v_q$ in A_t .)

Now, we estimate $\int_{A_t} |\nabla v_q|^p dx$ from below. For this, we apply Hölder's inequality and (13) to obtain

$$\left(\int_{A_t} (v_q - t) dx \right)^p \leq \left(\int_{A_t} (v_q - t)^{p^*} dx \right)^{\frac{p}{p^*}} |A_t|^{p - \frac{p}{p^*}} \leq \frac{|A_t|^{p - \frac{p}{p^*}}}{S_{N,p}^p} \int_{A_t} |\nabla v_q|^p dx.$$

Thus,

$$S_{N,p}^p |A_t|^{\frac{p}{p^*} - p} \left(\int_{A_t} (v_q - t) dx \right)^p \leq \int_{A_t} |\nabla v_q|^p dx$$

what yields, taking into account (16),

$$S_{N,p}^p |A_t|^{\frac{p}{p^*} - p} \left(\int_{A_t} (v_q - t) dx \right)^p \leq \lambda_p \|v_q\|_\infty^{q-1} \int_{A_t} (v_q - t) dx.$$

So we have

$$\left(\int_{A_t} (v_q - t) dx \right)^{p-1} \leq \frac{\lambda_p}{S_{N,p}^p} \|v_q\|_\infty^{q-1} |A_t|^{\frac{p+N(p-1)}{N}}.$$

This last inequality can be rewritten as

$$\left(\int_{A_t} (v_q - t) dx \right)^{\frac{N(p-1)}{p+N(p-1)}} \leq \left(\frac{\lambda_p}{S_{N,p}^p} \|v_q\|_\infty^{q-1} \right)^{\frac{N}{p+N(p-1)}} |A_t|. \quad (17)$$

By defining

$$f(t) := \int_{A_t} (v_q - t) dx,$$

from Cavalieri's Principle follows that

$$f(t) = \int_t^\infty |A_s| ds$$

and therefore $f'(t) = -|A_t|$. Thus, (17) can be rewritten as

$$1 \leq - \left(\frac{\lambda_p}{S_{N,p}^p} \|v_q\|_\infty^{q-1} \right)^{\frac{N}{p+N(p-1)}} f(t)^{-\frac{N(p-1)}{p+N(p-1)}} f'(t). \quad (18)$$

Integration of (18) yields

$$\begin{aligned} t &\leq K_{N,p} (\|v_q\|_\infty)^{\frac{N(q-1)}{p+N(p-1)}} \left[f(0)^{\frac{p}{p+N(p-1)}} - f(t)^{\frac{p}{p+N(p-1)}} \right] \\ &\leq K_{N,p} (\|v_q\|_\infty)^{\frac{N(q-1)}{p+N(p-1)}} (\|v_q\|_1)^{\frac{p}{p+N(p-1)}} \end{aligned}$$

where

$$K_{N,p} = \left(\frac{\lambda_p}{S_{N,p}^p} \right)^{\frac{N}{p+N(p-1)}} \left(\frac{p+N(p-1)}{p} \right).$$

Making $t \rightarrow \|v_q\|_\infty$ we obtain

$$\|v_q\|_\infty \leq K_{N,p} (\|v_q\|_\infty)^{\frac{N(q-1)}{p+N(p-1)}} (\|v_q\|_1)^{\frac{p}{p+N(p-1)}}$$

and hence, by noting that

$$0 \leq \frac{N(q-1)}{p+N(p-1)} < 1 \iff 1 \leq q < p \left(\frac{N+1}{N} \right)$$

we obtain

$$(\|v_q\|_\infty)^{\frac{p+N(p-q)}{p+N(p-1)}} \leq K_{N,p} (\|v_q\|_1)^{\frac{p}{p+N(p-1)}}$$

which gives (15). □

Lemma 3 *We have*

$$\lim_{q \rightarrow p^-} \left(\frac{\left(\int_\Omega e_p^q dx \right)^{\frac{p}{q}}}{\int_\Omega e_p^p dx} \right)^{\frac{1}{p-q}} = \lim_{q \rightarrow p^+} \left(\frac{\left(\int_\Omega e_p^q dx \right)^{\frac{p}{q}}}{\int_\Omega e_p^p dx} \right)^{\frac{1}{p-q}} = \theta_p \|e_p\|_p$$

where

$$\theta_p := \exp \left(\|e_p\|_p^{-p} \int_\Omega e_p |\ln e_p| dx \right). \quad (19)$$

Proof. It follows easily from L'Hospital rule, since

$$\left(\frac{\left(\int_\Omega e_p^q dx \right)^{\frac{p}{q}}}{\int_\Omega e_p^p dx} \right)^{\frac{1}{p-q}} = \exp \left(\frac{\frac{p}{q} \ln \left(\int_\Omega e_p^q dx \right) - \ln \int_\Omega e_p^p dx}{p-q} \right).$$

□

Theorem 4 *The asymptotic estimates for $\|u_q\|_q$ are true:*

$$0 < |\Omega|^{\frac{1}{p}-1} K_{N,p} \leq \liminf_{q \rightarrow p^+} \|u_q\|_q \quad (20)$$

and

$$\limsup_{q \rightarrow p^+} \|u_q\|_q \leq \theta_p \|e_p\|_p \leq \liminf_{q \rightarrow p^-} \|u_q\|_q. \quad (21)$$

Proof. If $1 < p < q < p^*$ we have

$$\int_\Omega u_q^p dx \leq \frac{1}{\lambda_p} \int_\Omega |\nabla u_q|^p dx = \int_\Omega u_q^q dx = \int_\Omega u_q^{q-p} u_q^p dx \leq \|u_q\|_\infty^{q-p} \int_\Omega u_q^p dx$$

showing that $\|u_q\|_\infty \geq 1$. Thus, (15) of Theorem 2 applied to $v_q = u_q$ and Hölder's inequality imply that

$$1 \leq \|u_q\|_\infty \leq (K_{N,p})^{\frac{p+N(p-1)}{p+N(p-q)}} \left(\|u_q\|_q |\Omega|^{1-\frac{1}{q}} \right)^{\frac{p}{p+N(p-q)}}$$

and hence

$$(K_{N,p})^{-\frac{p+N(p-1)}{p}} |\Omega|^{\frac{1}{q}-1} \leq \|u_q\|_q$$

which leads to the second inequality (20).

Both inequalities in (21) follow from Lemma 3, since

$$\lambda_q = R_q(w_q) < R_q(e_p) = \frac{\int_{\Omega} |\nabla e_p|^p dx}{\left(\int_{\Omega} e_p^q dx\right)^{\frac{p}{q}}} = \frac{\lambda_p \int_{\Omega} e_p^p dx}{\left(\int_{\Omega} e_p^q dx\right)^{\frac{p}{q}}}$$

gives

$$\|u_q\|_q = \left(\frac{\lambda_q}{\lambda_p}\right)^{\frac{1}{q-p}} < \left(\frac{\left(\int_{\Omega} e_p^q dx\right)^{\frac{p}{q}}}{\int_{\Omega} e_p^p dx}\right)^{\frac{1}{p-q}}, \text{ if } 1 < p < q$$

and

$$\|u_q\|_q = \left(\frac{\lambda_q}{\lambda_p}\right)^{\frac{1}{q-p}} > \left(\frac{\left(\int_{\Omega} e_p^q dx\right)^{\frac{p}{q}}}{\int_{\Omega} e_p^p dx}\right)^{\frac{1}{p-q}}, \text{ if } 1 < q < p < p^*.$$

□

Corollary 5 *The solution u_q defined by (8) satisfies*

$$0 < |\Omega|^{-1} K_{N,p} \leq \liminf_{q \rightarrow p} \|u_q\|_{\infty} \leq \limsup_{q \rightarrow p} \|u_q\|_{\infty} \leq (K_{N,p})^{\frac{p+N(p-1)}{p}} |\Omega|. \quad (22)$$

Proof. It follows from (20) and (21) of Theorem 4 that

$$\begin{aligned} |\Omega|^{-1} K_{N,p} &\leq \liminf_{q \rightarrow p^+} (|\Omega|^{-\frac{1}{q}} \|u_q\|_q) \leq \limsup_{q \rightarrow p^+} (|\Omega|^{-\frac{1}{q}} \|u_q\|_q) \\ &\leq |\Omega|^{-\frac{1}{p}} \theta_p \|e_p\|_p \leq \liminf_{q \rightarrow p^-} (|\Omega|^{-\frac{1}{q}} \|u_q\|_q). \end{aligned}$$

Since $|\Omega|^{-\frac{1}{q}} \|u_q\|_q \leq \|u_q\|_{\infty}$ we obtain

$$0 < |\Omega|^{-1} K_{N,p} \leq \liminf_{q \rightarrow p} (|\Omega|^{-\frac{1}{q}} \|u_q\|_q) \leq \liminf_{q \rightarrow p} \|u_q\|_{\infty}.$$

But (14) implies

$$\|u_q\|_1 \leq |\Omega|^{1-\frac{1}{q}} \|u_q\|_q \leq |\Omega|^{1-\frac{1}{q}} |\Omega|^{\frac{1}{q}} = |\Omega|. \quad (23)$$

So, it follows from Theorem 2 that

$$\|u_q\|_{\infty} \leq (K_{N,p})^{\frac{p+N(p-1)}{p+N(p-q)}} |\Omega|^{\frac{p}{p+N(p-q)}}, \quad (24)$$

what produces the last inequality in (22).

□

Lemma 6 For each $q \in (1, p^*)$, let $U_q := \frac{u_q}{\|u_q\|_\infty}$. Then U_q converges to e_p when $q \rightarrow p$ in $C^1(\overline{\Omega})$. Moreover,

$$\int_{\Omega} \frac{U_q^p - U_q^q}{q - p} dx \rightarrow \int_{\Omega} e_p^p |\ln e_p| dx \quad \text{as } q \rightarrow p. \quad (25)$$

Proof. Since

$$\begin{cases} -\Delta U_q &= \lambda_p \|u_q\|_\infty^{q-p} U_q^{q-1} & \text{in } \Omega, \\ U_q &= 0 & \text{on } \partial\Omega, \end{cases} \quad (26)$$

it follows from Corollary 5 that $\|u_q\|_\infty^{q-p} \rightarrow 1$ as $q \rightarrow p$. Because the right-hand side of the equation in (26) is uniformly bounded with respect to q close to p , we have that $\|U_q\|_{1,\alpha} \leq C$ for some positive constant C which is also uniform with respect to q close to p (see [14, Thm. 1]). Therefore, the compactness of the immersion $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ implies that, up to a subsequence, U_q converges in $C^1(\overline{\Omega})$ to a function $U \geq 0$ such that $\|U\|_\infty = 1$. Taking the limit $q \rightarrow p$ in the weak formulation (4) with $\lambda = \lambda_p \|u_q\|_\infty^{q-p}$, we obtain

$$\int_{\Omega} |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi dx = \lambda_p \int_{\Omega} |U|^{p-2} U \varphi dx$$

for an arbitrary test function $\varphi \in W_0^{1,p}(\Omega)$, proving that U is a nonnegative weak solution of (2). Since $\|U\|_\infty = 1$, we conclude that $U = e_p$ and the uniqueness of the limit proves that this convergence is in $C^1(\overline{\Omega})$.

In order to prove (25) we observe that

$$\left| \frac{U_q^p - U_q^q}{q - p} \right| \leq \frac{1}{|q - p|} \max_{0 \leq t \leq 1} |t^p - t^q| = \frac{1}{|q - p|} \frac{1}{p} \left(\frac{p}{q} \right)^{\frac{q}{q-p}} |q - p| = \frac{1}{p} \left(\frac{p}{q} \right)^{\frac{q}{q-p}},$$

what implies that $\frac{U_q^p - U_q^q}{q - p}$ is uniformly bounded with respect to q close to p , since

$$\limsup_{q \rightarrow p} \left| \frac{U_q^p - U_q^q}{q - p} \right| \leq \lim_{q \rightarrow p} \frac{1}{p} \left(\frac{p}{q} \right)^{\frac{q}{q-p}} = \frac{1}{p \exp(1)}.$$

Therefore (25) follows if we prove that

$$\frac{U_q^p - U_q^q}{q - p} \rightarrow e_p^p |\ln e_p| \quad \text{as } q \rightarrow p \quad \text{a.e. in } \Omega.$$

This convergence is uniform in $\mathcal{K} \subset \Omega$ compact. Indeed, if $0 < \epsilon < \min_{\mathcal{K}} e_p$, we have

$$0 < \min_{\mathcal{K}} e_p - \epsilon < e_p - \epsilon \leq U_q \leq e_p + \epsilon \quad \text{in } \mathcal{K}$$

for all q sufficiently close to p . Hence, in \mathcal{K} it is valid that

$$-(e_p + \epsilon)^q \ln(e_p + \epsilon) \leq \liminf_{q \rightarrow p^+} \frac{U_q^p - U_q^q}{q - p} \leq \limsup_{q \rightarrow p^+} \frac{U_q^p - U_q^q}{q - p} \leq -(e_p - \epsilon)^q \ln(e_p - \epsilon), \quad (27)$$

since

$$\lim_{q \rightarrow p^+} \frac{(e_p - \epsilon)^p - (e_p + \epsilon)^q}{q - p} = -(e_p + \epsilon)^p \ln(e_p + \epsilon)$$

and

$$\lim_{q \rightarrow p^+} \frac{(e_p + \epsilon)^p - (e_p - \epsilon)^q}{q - p} = -(e_p - \epsilon)^p \ln(e_p - \epsilon).$$

Therefore, making $\epsilon \rightarrow 0^+$ in (27) we conclude that

$$\lim_{q \rightarrow p^+} \frac{U_q^p - U_q^q}{q - p} = -e_p^p \ln e_p = e_p^p |\ln e_p|$$

and analogously

$$\lim_{q \rightarrow p^-} \frac{U_q^p - U_q^q}{q - p} = e_p^p |\ln e_p|.$$

□

Lemma 7 Suppose that $\|u_{q_n}\|_{q_n} \rightarrow L_{\pm}$ where $q_n \rightarrow p^{\pm}$. Then

$$\frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} \rightarrow \ln(L_{\pm}).$$

Proof. Let us first consider the case $q_n \rightarrow p^+$. Then, for each $\epsilon > 0$ we have

$$\frac{(L_+ - \epsilon)^{q_n-p} - 1}{q_n - p} \leq \frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} \leq \frac{(L_+ + \epsilon)^{q_n-p} - 1}{q_n - p}$$

for all n sufficiently large. Since

$$\ln(L_+ - \epsilon) = \lim_{q \rightarrow p^+} \frac{(L_+ - \epsilon)^{q-p} - 1}{q - p} \quad \text{and} \quad \ln(L_+ + \epsilon) = \lim_{q \rightarrow p^+} \frac{(L_+ + \epsilon)^{q-p} - 1}{q - p}$$

we obtain

$$\ln(L_+ - \epsilon) \leq \liminf \frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} \leq \limsup \frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} \leq \ln(L_+ + \epsilon).$$

Thus, by making $\epsilon \rightarrow 0^+$ we obtain

$$\frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} \rightarrow \ln L_+ \quad \text{as } q_n \rightarrow p^+$$

and analogously

$$\frac{\|u_{q_n}\|_{\infty}^{q_n-p} - 1}{q_n - p} = \ln L_- \quad \text{as } q_n \rightarrow p^-.$$

□

Theorem 8 *We have*

$$\limsup_{q \rightarrow p^-} \|u_q\|_\infty \leq \theta_p \leq \liminf_{q \rightarrow p^+} \|u_q\|_\infty. \quad (28)$$

Proof. It follows from Picone's identity (see [2, Thm 1.1]) that

$$|\nabla u|^p \geq |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\frac{u^p}{v^{p-1}} \right)$$

for all differentiable $u \geq 0$ and $v > 0$. Thus, by applying it to $U_q = \frac{u_q}{\|u_q\|_\infty}$ and e_p we obtain

$$\int_\Omega |\nabla U_q|^p dx \geq \int_\Omega |\nabla e_p|^{p-2} \nabla e_p \cdot \nabla \left(\frac{U_q^p}{e_p^{p-1}} \right) dx. \quad (29)$$

(A consequence of Hopf's boundary lemma (see [19]) is that $U_q^p / e_p^{p-1} \in W_0^{1,p}(\Omega)$.)

Since U_q satisfies (26) and (λ_p, e_p) satisfies (2) we obtain from (29) that

$$\lambda_p \|u_q\|_\infty^{q-p} \int_\Omega U_q^q dx \geq \lambda_p \int_\Omega \frac{U_q^p}{e_p^{p-1}} e_p^{p-1} dx,$$

that is,

$$\|u_q\|_\infty^{q-p} \int_\Omega U_q^q dx \geq \int_\Omega U_q^p dx. \quad (30)$$

Let us assume that $p < q$. In this case we rewrite (30) as

$$\frac{\|u_q\|_\infty^{q-p} - 1}{q-p} \int_\Omega U_q^q dx \geq \int_\Omega \frac{U_q^p - U_q^q}{q-p} dx. \quad (31)$$

Now, if $q_n \rightarrow p^+$ is such that $\lim_{q_n \rightarrow p^+} \|u_{q_n}\|_\infty = L_+$ then it follows from Lemma 6 that

$$\int_\Omega U_{q_n}^{q_n} dx \rightarrow \int_\Omega e_p^p dx \quad \text{and} \quad \int_\Omega \frac{U_{q_n}^p - U_{q_n}^{q_n}}{q_n - p} dx \rightarrow e_p |\ln e_p|.$$

Therefore, Lemma 7 and (31) give

$$\ln L_+ \geq \frac{\int_\Omega e_p^p |\ln e_p| dx}{\int_\Omega e_p^p dx}$$

or equivalently

$$\theta_p = \exp \left(\frac{\int_\Omega e_p^p |\ln e_p| dx}{\int_\Omega e_p^p dx} \right) \leq L_+.$$

We conclude that

$$\theta_p \leq \liminf_{q \rightarrow p^+} \|u_q\|_\infty.$$

The case $q \rightarrow p^-$ is analogous by rewriting (30) as

$$\frac{\|u_q\|_\infty^{q-p} - 1}{p-q} \int_\Omega U_q^q dx \leq \int_\Omega U_q^p \frac{1 - U_q^{q-p}}{p-q} dx.$$

We obtain

$$L_- \leq \theta_p = \exp \left(\frac{\int_{\Omega} e_p^p |\ln e_p| dx}{\int_{\Omega} e_p^p dx} \right),$$

where $L_- = \lim_n \|u_{q_n}\|_{\infty}$ and $q_n \rightarrow p^-$. Therefore we conclude that

$$\limsup_{q \rightarrow p^-} \|u_q\|_{\infty} \leq \theta_p.$$

□

Now we prove our main result on the asymptotic behavior of the resonant Lane-Emden problem (6).

Theorem 9 *We have the convergence in $C^1(\overline{\Omega})$*

$$\lim_{q \rightarrow p} u_q = e_p \exp \left(\|e_p\|_p^{-p} \int_{\Omega} e_p |\ln e_p| dx \right)$$

Proof. The asymptotic estimate (22) of Theorem 2 combined with classical results ([8, 14, 18]) guarantee that u_q is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ (with respect to q) for all q close enough to p , for some $0 < \alpha < 1$. Hence we conclude, as in the proof of Lemma 6, that up to a subsequence, u_q converges in $C^1(\overline{\Omega})$, when $q \rightarrow p^+$, to a weak solution $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ of the resonant Lane-Emden problem (2) such that $u = k_+ e_p$.

It follows from (21) of Theorem 4 that $k_+ \leq \theta_p$. On the other hand, (28) yields $\theta_p \leq k_+$, proving that $k_+ = \theta_p$. The uniqueness of the limit guarantees that

$$\lim_{q \rightarrow p^+} u_q = \theta_p e_p$$

in $C^1(\overline{\Omega})$.

Analogously, u_q converges in $C^1(\overline{\Omega})$ to $\theta_p e_p$ if $q \rightarrow p^-$ in $C^1(\overline{\Omega})$.

□

Corollary 10 *The application $q \rightarrow \lambda_q$ is differentiable at $q = p$ and*

$$\frac{d}{dq} [\lambda_q]_{q=p} = \lambda_p \ln(\theta_p \|e_p\|_p). \quad (32)$$

Proof. Since $\|u_q\|_q = \left(\frac{\lambda_q}{\lambda_p} \right)^{\frac{1}{q-p}}$ it follows from Theorem 9 that

$$\lim_{q \rightarrow p} \ln \left(\frac{\lambda_q}{\lambda_p} \right)^{\frac{1}{q-p}} = \lim_{q \rightarrow p} \frac{\ln \lambda_q - \ln \lambda_p}{q - p} = \ln(\theta_p \|e_p\|_p).$$

Thus $\ln \lambda_q$ is differentiable at $q = p$ and $\frac{d}{dq} [\ln \lambda_q]_{q=p} = \ln(\theta_p \|e_p\|_p)$ which is equivalent to differentiability of λ_q at $q = p$ with $\frac{d}{dq} [\lambda_q]_{q=p}$ given by (32). □

Remark 11 *The estimate*

$$(\lambda_p \|\xi_p\|_\infty)^{-1} \leq \liminf_{q \rightarrow p^-} \|u_q\|_q^q$$

where ξ_p is the first eigenfunction normalized by the $W_0^{1,p}$ norm, was proved in [3].

Since $\liminf_{q \rightarrow p^-} \|u_q\|_q^q = (\theta_p \|e_p\|_p)^p$ (as proved in Theorem 9) and $(\lambda_p \|\xi_p\|_\infty)^{-1} = \|e_p\|_p^p < (\theta_p \|e_p\|_p)^p$, we see that this estimate is not sharp.

It is interesting to remark that in the Laplacian case $p = 2$, a simpler argument can be used to prove directly that

$$\lim_{q \rightarrow 2} u_q = e_2 \exp \left(\|e_2\|_2^{-2} \int_\Omega e_2 |\ln e_2| dx \right).$$

Such argument has already appeared in [7], where the asymptotic behavior of positive solutions of a logistical type problem for the Laplacian was studied. It explores a self-adjointness property type which is valid for the Laplacian. In fact,

$$\lambda_2 \|u_q\|_\infty^{q-2} \int_\Omega U_q^{q-1} e_2 dx = \int_\Omega \nabla U \cdot \nabla e_2 dx = \int_\Omega \nabla e_2 \cdot \nabla U_2 dx = \lambda_2 \int_\Omega e_2 U_q dx$$

leads to

$$\frac{\|u_q\|_\infty^{q-2} - 1}{q-2} \int_\Omega U_q^{q-1} e_2 dx = \int_\Omega \left(\frac{1 - U_q^{q-2}}{q-2} \right) e_2 U_q dx.$$

Thus, if $\|u_{q_n}\|_\infty \rightarrow L$ then

$$\ln L = \lim_n \ln \frac{\|u_{q_n}\|_\infty^{q-2} - 1}{q_n - p} = \lim_n \frac{\int_\Omega \left(\frac{1 - U_{q_n}^{q_n-2}}{q_n-2} \right) e_2 U_{q_n} dx}{\int_\Omega U_{q_n}^{q_n-1} e_2 dx} = \|e_2\|_2^{-2} \int_\Omega e_2 |\ln e_2| dx$$

proving that $\|u_q\|_\infty \rightarrow \exp \left(\|e_2\|_2^{-2} \int_\Omega e_2 |\ln e_2| dx \right)$.

Order of growth analysis was introduced in [10] in order to extend this results to the p -Laplacian but the arguments given there are not clear. Our approach avoids this method by using Picone's identity and is simpler.

2.1 The first eigenpair (λ_p, e_p) via non-resonant problems

In this section we show how to build sequences converging to the first eigenpair of the p -Laplacian from positive solutions (ground states in the super-linear case) of the non-resonant Lane-Emden problem

$$\begin{cases} -\Delta u &= \lambda |u|^{q-2} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \quad (33)$$

with $\lambda > 0$ arbitrarily fixed and $q \rightarrow p$. As mentioned before, the advantage here is that λ can be chosen arbitrarily in computational implementations of (33) and does not need to be close to λ_p . A numerical solution of the nonlinear problem (33) is easier to obtain than directly compute the first eigenvalue of the p -Laplacian.

For each $1 < q < p^*$ we consider the weak solution of (33) given by

$$u_{\lambda,q} := \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} u_q \quad (34)$$

where u_q is the positive solution of the resonant Lane-Emden problem (6).

Theorem 12 For any $\lambda > 0$ we have

$$\lim_{q \rightarrow p} (\lambda \|u_{\lambda,q}\|_{\infty}^{q-p}) = \lambda_p$$

and

$$\frac{u_{\lambda,q}}{\|u_{\lambda,q}\|_{\infty}} \rightarrow e_p \text{ in } C^1(\Omega) \text{ as } q \rightarrow p.$$

Proof. This follows from Lemma 6 combined with (34) since

$$\frac{u_{\lambda,q}}{\|u_{\lambda,q}\|_{\infty}} = U_q \text{ and } \lambda \|u_{\lambda,q}\|_{\infty}^{q-p} = \lambda_p \|u_q\|_{\infty}^{q-p}.$$

□

Therefore, one can compute a numerical solution of problem (33) with q close to p in order to obtain approximations for λ_p and e_p simultaneously. This approach is used in [5], where the iterative sub- and super-solution method in the sub-linear case is also applied.

Remark 13 Theorem 12 remains true if one replaces the L^{∞} norm by the L^q norm. That is, one also has the convergences

$$\lambda \|u_{\lambda,q}\|_q^{q-p} \rightarrow \lambda_p \text{ and } \frac{u_{\lambda,q}}{\|u_{\lambda,q}\|_q} \rightarrow \frac{e_p}{\|e_p\|_p} \text{ in the } C^1 \text{ norm.}$$

We now show that the convergence $\mu_q \rightarrow \lambda_q$ implies that the quotient

$$\Lambda_q := \lambda \frac{\|u_{\lambda,q}\|_q^q}{\|u_{\lambda,q}\|_p^p} = \lambda \frac{(\frac{\lambda}{\lambda_p})^{\frac{q}{p-q}} \|u_q\|_q^q}{(\frac{\lambda}{\lambda_p})^{\frac{p}{p-q}} \|u_q\|_p^p} = \lambda_p \frac{\|u_q\|_q^q}{\|u_q\|_p^p},$$

also converges to λ_p as $q \rightarrow p$. Moreover, we estimate the convergence order in the approximation of Λ_q by μ_q .

Theorem 14 There holds:

(i) $\lambda_p \leq \Lambda_q$.

(ii) $\lim_{q \rightarrow p} \Lambda_q = \lambda_p$.

(iii) If q is close enough to p , then

$$\left| \frac{\Lambda_q}{\mu_q} - 1 \right| \leq K |q - p|$$

for some positive constant K which does not depends on q .

Proof. Since $\lim_{q \rightarrow p} \|U_q\|_q^q = \lim_{q \rightarrow p} \|U_q\|_p^p = \|e_p\|_p^p$, we have

$$\lambda_p \leq \frac{\|\nabla u_{\lambda,q}\|_p^p}{\|u_{\lambda,q}\|_p^p} = \frac{\lambda \|u_{\lambda,q}\|_q^q}{\|u_{\lambda,q}\|_p^p} = \Lambda_q = \lambda \|u_{\lambda,q}\|_{\infty}^{q-p} \frac{\|U_q\|_q^q}{\|U_q\|_p^p} = \mu_q \frac{\|U_q\|_q^q}{\|U_q\|_p^p}$$

proving (i) and (ii), since

$$\lambda_p \leq \lim_{q \rightarrow p} \Lambda_q = \lim_{q \rightarrow p} \mu_q \frac{\|U_q\|_q^q}{\|U_q\|_p^p} = \left(\lim_{q \rightarrow p} \mu_q \right) \left(\lim_{q \rightarrow p^-} \frac{\|U_q\|_q^q}{\|U_q\|_p^p} \right) = \lambda_p.$$

We also have

$$\left| \frac{\Lambda_q}{\mu_q} - 1 \right| = \left| \frac{\|U_q\|_q^q}{\|U_q\|_p^p} - 1 \right| = \frac{1}{\|U_q\|_p^p} \left| \int_{\Omega} (U_q^q - U_q^p) dx \right| \leq \frac{1}{\|U_q\|_p^p} \int_{\Omega} \max_{0 \leq t \leq 1} |t^q - t^p| dx \leq K |p - q|,$$

since $\|U_q\|_p^p \rightarrow \|e_p\|_p^p$ uniformly. □

Remark 15 Since $\mu_q \leq \lambda_p$ in the sub-linear case (see [5]), the rate of convergence of both μ_q and Λ_q to λ_p is at least $O(p - q)$.

3 Asymptotic behavior for the non-resonant problem

In this section we describe the asymptotic behavior of $u_{\lambda,q}$ in the C^1 norm when $q \rightarrow p$, for each $\lambda > 0$. Our results generalize that in [3] (where the sub-linear case was handled in $W_0^{1,p}$) and in [11] (where the super-linear case was handled in the same space). Our method is independent of those already used.

Theorem 16 Let $u_{\lambda,q}$ be the positive solution of (33) defined by (34). Then

$$\lim_{q \rightarrow p^-} \|u_{\lambda,q}\|_{C^1} = \begin{cases} 0 & \text{if } \lambda < \lambda_p \\ \infty & \text{if } \lambda > \lambda_p \end{cases} \quad \text{and} \quad \lim_{q \rightarrow p^+} \|u_{\lambda,q}\|_{C^1} = \begin{cases} \infty & \text{if } \lambda < \lambda_p \\ 0 & \text{if } \lambda > \lambda_p \end{cases}$$

Proof. From Theorem 4 follows that

$$\|u_{\lambda,q}\|_{C^1} = \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} \|u_q\|_{C^1} \geq \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} \|u_q\|_{\infty} \geq \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} |\Omega|^{-\frac{1}{q}} \|u_q\|_q \geq \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} |\Omega|^{-\frac{1}{q}} \beta,$$

for some positive constant β which does not depend on q close to p . Thus, when $q \rightarrow p^-$ and $\lambda > \lambda_p$ or when $q \rightarrow p^+$ and $\lambda < \lambda_p$, then $\|u_{\lambda,q}\|_{C^1} \rightarrow \infty$.

Since u_q is uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$ with respect to q if q is close to p , the continuity of the immersion $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ implies that $\|u_q\|_{C^1} \leq K$ for some positive constant K that does not depend on q close to p . Hence, when $q \rightarrow p^-$ and $\lambda < \lambda_p$ or when $q \rightarrow p^+$ and $\lambda > \lambda_p$, we have

$$\|u_{\lambda,q}\|_{C^1} = \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} \|u_q\|_{C^1} \leq K \left(\frac{\lambda}{\lambda_p} \right)^{\frac{1}{p-q}} \rightarrow 0. \quad \square$$

We remark that, if $\{v_{\lambda,q}\}_{p < q < p^*}$ is another family of positive weak solutions of (1), then we have $\|v_{\lambda,q}\|_q \rightarrow \infty$ when $q \rightarrow p^-$ and $\lambda > \lambda_p$ or when $q \rightarrow p^+$ and $\lambda < \lambda_p$.

In fact, it follows from (9) and Theorem 2 that

$$\|v_{\lambda,q}\|_q > \|u_{\lambda,q}\|_q = \left(\frac{\lambda}{\lambda_p}\right)^{\frac{1}{p-q}} \|u_q\|_q \geq M \left(\frac{\lambda}{\lambda_p}\right)^{\frac{1}{p-q}}$$

for some positive constant M which is uniform with respect to q , for all q close to p .

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